Local stability threshold in del Pezzo surface of degree 2

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Definition

Let $x_0, ..., x_n$ be affine coordinates on $\mathbb{A}^{n+1}$ and let the group $\mathbb{C}^*$ act via:

$$\lambda(x_0, ..., x_n) = (\lambda^{a_0} x_0, ..., \lambda^{a_n} x_n).$$

Then, the **weighted projective space** $\mathbb{P}(a_0, ..., a_n)$ is the quotient $(\mathbb{A}^{n+1} \setminus 0)/\mathbb{C}^*$. Under this group action $x_0, ..., x_n$ are the **homogeneous coordenates** on $\mathbb{P}(a_0, ..., a_n)$.

Notice that if $a_0 = ... = a_n = 1$ we get the ordinary projective space $\mathbb{P}^n$.

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Definition

A polynomial f is **weighted homogeneous** of degree d if every monomial x^α appearing in f satisfies $\alpha \cdot (a_0, ..., a_n) = d$. Then $f = 0$ is well defined on $\mathbb{P}(a_0, ..., a_n)$ when f is weighted homogeneous so that one can define varieties in $\mathbb{P}(a_0, ..., a_n)$ using weighted homogeneous ideals of $\mathbb{C}[x_0, ..., x_n]$.

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Definition

A **projective variety** is an algebraic variety that is a subset of one of the projective spaces defined above.

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For X surface we can check ampleness as follows:

Theorem (Nakai/Moishezon Criterion for Ampleness)

A line bundle L on a nonsingular projective surface X is ample if

$$L^2 > 0 \text{ and } L \cdot C > 0 \text{ for any curve } C \subset X.$$

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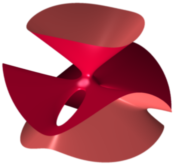
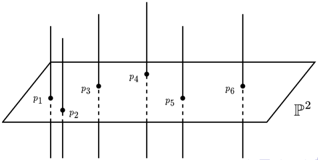
Examples:

- $\mathbb{P}^2$
- Any cubic surface.





3. Del Pezzo surfaces.

Let X be a del Pezzo surface of degree 2. Then $X \subseteq \mathbb{P}(1,1,1,2)$ (with homogeneous coordinates x, y, z, w) can be given by the homogeneous equation,

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Example:

4. Introduction to K-stability.

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Naive explanation of moduli: A (fine) moduli space requires families of points on the moduli space to correspond to families of the kind of object you're trying to classify (in a very precise way).

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where

$$\tau(E) = \sup\{t \in \mathbb{Q} | \mathrm{vol}(f^*(-K_X) - tE) > 0\}$$

is called the pseudoeffective threshold.

4. Introduction to K-stability.

δ-invariant (stability threshold):

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Note that,

K-stability $\Rightarrow$ K-polystability $\Rightarrow$ K-semistability

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References: K-stability survey in Chenyang Xu's webpage and The Calabi book by Araujo, Castrevet, Chelchov, Fujita, Kaloghinos, Martinez-Garcia, Shramov, Suss, Viswanathan.

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For any Fano variety X of dimension greater or equal to 3, $\delta_p(X)$ is rational for every $p \in X$.

Theorem (EA '23)

Let $X \subseteq \mathbb{P}(1:1:1:2)$ be a smooth del Pezzo surface of degree 2 and let $p_0 = (x_0, y_0, z_0, w_0) \in X$ be a closed point with $w_0 \neq 0$. Let $Q = T_{p_0}(X) \cap X$ be the tangent hyperplane section. Assume $Q = L \cup C$ where L is a line and C is a cubic curve that intersects transversally at p_0. Then $\delta_{p_0}(X) = \frac{6}{71}(11 + 8\sqrt{3})$.

THANK YOU!